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Reaction-Convection in Incompressible 3D-Fluid: A Homogenization Problem

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We consider reaction-convection in a stationary incompressible flow which is close to a planar motion. Under certain conditions, we introduce the notion of relative entropy for such a deterministic flow to describe the motion of the spot occupied by an ingredient.

KEY WORDS: Reaction-Diffusion; Incompressible fluid; Homogenization; Relative entropy.

1. STATEMENT OF THE PROBLEM

We will consider a stationary motion of incompressible 3-D fluid, which is close to a planar motion (see [5]). As is known, 2D-incompressible fluid can be described by the stream function $\psi(x_1, x_2)$. The velocity field of such a fluid $V(x_1, x_2) = \bar{\nabla}\psi(x_1, x_2) = (-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1})$.

Consider now 3D-fluid with the velocity

$$V^{\varepsilon}(x) = \bar{\nabla}\psi(x) + \varepsilon b(x), \quad x \in R^3.$$

Here $\bar{\nabla}\psi(x) = (-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}, 0)$, $div\, b(x) = 0$, $0 < \varepsilon << 1$. We assume that $b(x) = (b_1(x), b_2(x), b_3(x))$ is continuously differentiable and derivatives are bounded uniformly in $x \in R^3$. Moreover, let $b_3(x) \geq \beta > 0$, and $b_3(x_1, x_2, x_3) \equiv b_3(x_1, x_2, x_3 + 2\pi)$.

It is clear that the motion with velocity $V^{\varepsilon}(x)$, on any finite time interval $[0, T]$, is close to the planar motion as $\varepsilon << 1$. To observe the displacements of order 1 as $\varepsilon \downarrow 0$, one should rescale the time $t \to \frac{t}{\varepsilon}$.

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If an ingredient is put in the fluid which moves together with the fluid and takes part in a chemical reaction, the evolution of the ingredient density, after an appropriate time rescaling, is described by the equation

$$\frac{\partial u^{\varepsilon}(t,x)}{\partial t} = \frac{1}{\varepsilon}\bar{\nabla}\psi(x)\cdot\nabla u^{\varepsilon} + b(x)\cdot\nabla u^{\varepsilon} + f(u^{\varepsilon}) = L^{\varepsilon}u^{\varepsilon} + f(u^{\varepsilon}), \quad (1)$$
$$u^{\varepsilon}(0,x) = g(x) \geq 0, \quad Supp\ g = G_0$$

The function $f(u)$ in (1) describes the reaction, $g(x)$ is the initial density of the ingredient. Let $f(u)$ be of the Kolmogorov-Petrovskii-Piskunov (KPP) type: $f(u) = c(u)u$, $c(u) < 0$ for $u > 1$ and $c(u) > 0$ for $u < 1$, $c(0) = c = \max c(u)$; $c(u)$ is assumed to be Lipschitz continuous.

The stream function $\psi(x_1, x_2)$ is assumed to be smooth enough, say, has three continuous derivatives, $\lim_{|x|\to\infty} \psi(x) = \infty$ and $\psi(x)$ is generic. The latter means that $\psi(x)$ has a finite number of critical points, and each of them is non-degenerate. Let, for brevity, $\psi(x)$ have different values at different critical points.

One can expect that, because of the periodicity of $b_3(x_1, x_2, x_3)$ in x_3, a homogenization effect in the motion along x_3-axis will appear: a constant asymptotic speed of the flow in x_3-direction will be established for large distances and small ε; the propagation of the ingredient has an asymptotic speed.

It worth mentioning that the reaction term, say, of KPP type in a first order equation does not change the area where the ingredient is situated at a time t. It just changes the value of the ingredient's density in the area where particles are brought by the flow. So that, for each $\varepsilon > 0$, a certain domain exists where the ingredient situated at time t (where the solution $u^{\varepsilon}(t,x)$ of problem (1) is positive). But this domain is, very sensitive to changes of ε, and, in general, no limit of $u^{\varepsilon}(t,x)$ exists as $\varepsilon \downarrow 0$.

On the other hand small random perturbations are always available in the system. So that a more realistic model should include not just motion of the ingredient particles together with the flow, but also a small diffusion term, then equation (1) should be replaced by the following:

$$\frac{\partial u^{\varepsilon,\kappa}(t,x)}{\partial t} = \frac{\kappa}{2}\sum a_{ij}(x)\frac{\partial^2 u^{\varepsilon,\kappa}}{\partial x_i \partial x_j} + \frac{1}{\varepsilon}\bar{\nabla}\psi(x)\cdot\nabla u^{\varepsilon,\kappa} + b\cdot\nabla u^{\varepsilon,\kappa} + f(u^{\varepsilon,\kappa})$$
$$= L^{\varepsilon,\kappa}u^{\varepsilon,\kappa} + f(u^{\varepsilon,\kappa}), \qquad u^{\varepsilon,\kappa}(0,x) = g(x). \quad (2)$$

Here $(a_{ij}(x))$ is the diffusion matrix, which for simplicity is considered diagonal and independent of x; $0 < \kappa << 1$.

It turns out that the double limit, first, as $\varepsilon \downarrow 0$ and then as $\kappa \downarrow 0$, of $u^{\varepsilon,\kappa}(t,x)$, under some natural assumptions, exists, and one can calculate the effective speed of the flow and of the ingredient in x_3-direction. This limit turns out independent of the diffusion matrix $(a_{ij}(x))$ (assuming that it is non-degenerate). So that the effective velocities should be considered as characteristics of the reaction-convection problem. The diffusion term is used just for a regularization (compare with [1], [5]).

The diffusion process $X_t^{\varepsilon,\kappa}$ corresponding to the operator $L^{\varepsilon,\kappa}$ in (2) has a fast and a slow components as $\varepsilon \downarrow 0$. The fast component which, roughly speaking, coincides with the basic planar motion, can be characterized by the invariant measure of the Hamiltonian system $\dot{X} = \bar{\nabla}\psi(x)$ on the level set component of the stream function $\psi(x)$. At each $t > 0$, this level set component is defined by the position of the slow component of $X_t^{\varepsilon,\kappa}$. This slow component is the projection $Y(X_t^{\varepsilon,\kappa})$ of $X_t^{\varepsilon,\kappa}$ on a space Π, which is the product of a graph Γ related to $\psi(x)$ and R^1 (such spaces are called open book) (see [6]).

The process $Y_t^{\varepsilon,\kappa} = Y(X_t^{\varepsilon,\kappa})$ on Π converges weakly in the space of continuous function $\varphi : [0, T] \to \Pi$, $T < \infty$, as, first, $\varepsilon \downarrow 0$, and then $\kappa \downarrow 0$, to a stochastic process Y_t on Π. The process Y_t is deterministic inside each page of Π and has stochastic behavior just when Y_t comes to the binding of the open book Π.

It is important to note that the limiting process Y_t on Π is the same for different diffusion matrices (a_{ij}). So that the stochasticity of Y_t is an intrinsic property of the deterministic system [5]. The independence of the asymptotic speed (of the interface between the areas where the density of the ingredient is close to 1 and to zero) of the matrix (a_{ij}) is a manifestation of the fact that the process Y_t is independent of (a_{ij}).

If the stream function has, just one minimum, the graph Γ consists of one half line $I = \{z : z \geq \min_{x \in R^2} \psi(x)\}$ and $\Pi = \mathbf{I} \times R^1$ has just one page. The limiting process Y_t, in this case, has no stochasticity, and the reaction term does not change the evolution of the area occupied by the ingredient as $\varepsilon, \kappa \downarrow 0$. We consider this case in the next section.

In section 3, we study the case of stream functions with saddle points. Then the limiting slow motion Y_t is stochastic, and the asymptotic speed in x_3-direction is defined by the law of large numbers. Central limit theorem can be used to describe the deviations from the motion with effective speed.

The evolution of the area occupied by the ingredient in the case of stream function with saddle points is defined by the large deviations from typical behavior of Y_t. These large deviations are characterized by the relative entropy. We calculate the speed of the ingredient propagation under certain symmetry assumptions. We also give a sketch of the result in general situation.

In the last section 4, we study the behavior of ingredient on large time intervals when $t \sim \kappa^{-1/2}$. In this time scale, the motion of the spot occupied by the ingredient depends on the diffusion matrix (a_{ij}). We describe this motion using large deviation estimates for processes with a small diffusion.

2. NO-SADDLE-POINTS CASE

Let $\psi(x), x \in R^2$, have just one minimum at O. Then the trajectories of corresponding 2D-flow are periodic and each of them belongs to a level set of $\psi(x)$. Put $C(z) = \{x \in R^2 : \psi(x) = z\}$; let $G(z)$ be the domain bounded by $C(z)$.

The period of rotation along $C(z)$ denote by $T(z)$:

$$T(z) = \oint_{C(z)} \frac{dl}{|\nabla\psi(x)|},$$

where dl is the length element on $C(z)$. The invariant density of the flow on $C(z)$ is equal to $m_z(x) = (T(z)|\nabla\psi(x)|)^{-1}$, $x \in C(z)$.

Let $b_{1,2}(x) = (b_1(x), b_2(x))$, $x_{1,2} = (x_1, x_2)$. Then the system governed by the operator L^ε from (1) has the form:

$$\dot{X}^\varepsilon_{1,2}(t) = \frac{1}{\varepsilon}\bar{\nabla}\psi(X^\varepsilon_{1,2}) + b_{1,2}(X^\varepsilon_{1,2}, X^\varepsilon_3), \quad \dot{X}^\varepsilon_3(t) = b_3(X^\varepsilon_{1,2}, X^\varepsilon_3) \tag{3}$$

This motion has a fast component, which is, actually, the motion along the non-perturbed trajectory $\dot{X}^\varepsilon_{1,2} = \frac{1}{\varepsilon}\bar{\nabla}\psi(X^\varepsilon_{1,2})$, and a two-dimensional slow component, which can be described by the evolution of $\psi(X^\varepsilon_{1,2}(t)) = \psi^\varepsilon_t$ and of $X^\varepsilon_3(t)$. Taking into account that $\nabla\psi(x)\cdot\bar{\nabla}\psi(x) = 0$, we have:

$$\psi^\varepsilon_t - \psi^\varepsilon_0 = \int_0^t \nabla\psi(X^\varepsilon_{1,2}(s))\cdot b_{1,2}(X^\varepsilon_{1,2}(s), X^\varepsilon_3(s))ds,$$

$$X^\varepsilon_3(t) - X^\varepsilon_3(0) = \int_0^t b_3(X^\varepsilon_{1,2}(s), X^\varepsilon_3(s))ds.$$

Standard averaging principle implies (see references in section 7.1 of [6]) that the slow component $(\psi^\varepsilon_t, X^\varepsilon_3(t))$ converges uniformly on any finite time interval $[0, T]$ as $\varepsilon \downarrow 0$ to $(Z(t), Y(t))$, where $(Z(t), Y(t))$ is the solution of the system

$$\begin{aligned}\dot{Z}(t) &= \frac{1}{T(Z(t))}\oint_{C(z(t))} \frac{\nabla\psi(v)\cdot b_{1,2}(v)}{|\nabla\psi(v)|}dl,\\ \dot{Y}(t) &= \frac{1}{T(Z(t))}\oint_{C(z(t))} \frac{b_3(v)}{|\nabla\psi(v)|}dl\end{aligned} \tag{4}$$

Put

$$A(z, y) = \int_{G(z)} b_3(x_1, x_2, y)dx_1dx_2.$$

Using the divergence theorem, one can see that

$$\oint_{C(z)} \frac{\nabla\psi(v)\cdot b_{1,2}(v, y)}{|\nabla\psi(v)|}dl = \int_{G(z)} div_{1,2}b_{1,2}(x_{1,2}, y)dx_1dx_2,$$

where $div_{1,2}b_{1,2}(x_1, x_2, y) = \frac{\partial b_1}{\partial x_1}(x_{1,2}, y) + \frac{\partial b_2}{\partial x_2}(x_{1,2}, y)$. Taking into account that $divb(X) = 0$, we can deduce that $div_{1,2}b_{1,2}(x_{1,2}, y) = -\frac{\partial b_3}{\partial y}(x_{1,2}, y)$, then the right hand side of the first of equations (4) is equal to

$$-\frac{1}{T(z)}\frac{\partial A(z, y)}{\partial y}.$$

One can check that

$$\oint_{C(z)} \frac{b_3(x_{1,2}, y)}{|\nabla \psi(x_{1,2})|} dl = \frac{\partial}{\partial z} A(z, y)$$

So that system (4) can be written as follows

$$\begin{cases} \dot{Z}_t = -\frac{1}{T(Z_t)} A_y'(Z_t, y_t) \\ \dot{y}_t = \frac{1}{T(Z_t)} A_z'(Z_t, y_t) \end{cases} \tag{5}$$

The function $A(z, y)$ is a first integral of system (5). Because of our assumptions on $b_3(x)$,

$$A(z, y) > 0, \quad A_z'(z, y) > 0, \quad A(z, y + 2\pi) \equiv A(z, y), \quad \lim_{z \to \infty} A(z, y) = \infty. \tag{6}$$

Consider the graph of $A(z, y)$. Since the function is 2π-periodic in y, we draw it over the cylinder (Fig. 1) of radius 1. Put $m = A(\psi_0, x_3(0))$. Then $A(\psi_t, x_3(t)) = m$ for all $t \geq 0$. The solution of system (5) is periodic in t and goes along the projection $\sum_m = \{(z, y) : A(z, y) = m\}$ of the m-level set of $A(z, y)$ over the cylinder (Fig. 1). One rotation along this curve $\sum_m$ takes time

$$\tau_m = \oint_{\sum_m} \frac{T(z) dl}{|\nabla A(z, y)|}.$$

Thus the asymptotic speed as $\varepsilon \downarrow 0$ and $t >> 1$ is equal to $\nu_m = \frac{2\pi}{\tau_m}$. The reaction term in this case will not change the speed. It can just change the density of the ingredient particles.

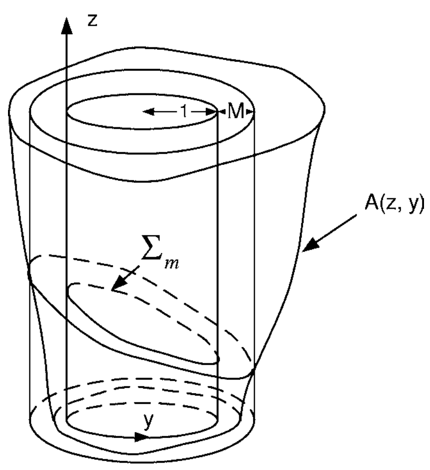

Fig. 1.

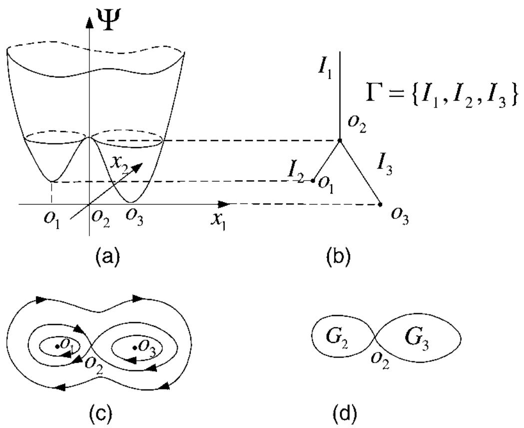

Fig. 2.

One should note that the domain occupied by ingredient for $t > 0$ and $\varepsilon << 1$, in general, is rather irregular: since the flow is incompressible, the volume of the domain $\{x \in R^3 : u^\varepsilon(t, x) > 0\}$ does not change, and, under generic assumptions, the domain becomes more and more stretched as $\varepsilon \downarrow 0$.

We will see that such a relatively simple expression for the time τ_m needed for an ingredient particle to pass one period in the x_3-direction is available, roughly speaking, just in the case, when the stream function has no saddle points. In the next section, we consider stream functions with one or more saddle points. Then, if the corresponding $\sum_m$-curve cross the level H^* of a saddle point (if $A(H^*, y) = A(\psi(x_1(0), x_2(0)), x_3(0))$ for some $y \in [0, 2\pi]$), the asymptotic rotation time becomes, in a sense, random, and the asymptotic speed is defined by a law of large numbers.

3. MANY CRITICAL POINTS

Suppose now that the stream function $\psi(x_!, x_2)$ of the basic planar motion has saddle points. Let, for brevity, $\psi(x)$ have just one saddle point O_2 (Fig. 2), and $\psi(O_2) = H^*$.

The level set $C(z) = \{(x_1, x_2) \in R^2 : \psi(x_1, x_2) = z\}$, in general, consists of several connected components $C_k(z)$: $C(z) = \cup_{k=1}^{n(z)} C_k(z)$. Let Γ be the graph homeomorphic to the set of all connected components of the level sets of $\psi(x)$ (Fig. 2b) (compare with [6], [4]). For each $C_k(z)$ one can introduce the period $T_k(z)$ of rotation along $C_k(z) : T_k(z) = \oint_{C_k(z)} |\nabla\psi(x)|^{-1} dl$. We denote by $G_k(z)$ the domain in R^2 bounded by $C_k(z)$. A ∞-shaped curve $\gamma(O_2) = \gamma = \{x \in R^2 : \psi(x) = \psi(O_2)\}$ is related to each saddle point O; $\gamma = \gamma(O_2)$ is shown in Fig. 2. Let G_2 and G_3 be the domains bounded by this curve (Fig 2^d).

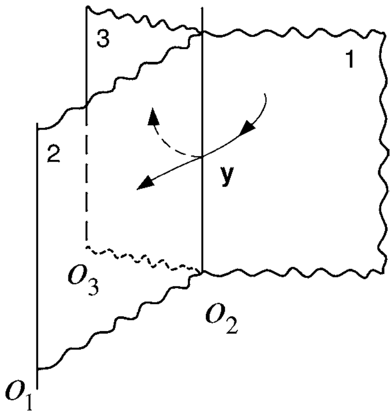

Fig. 3.

Let all the edges of Γ be numbered. Each point $y \in \Gamma$ is defined in the unique way by the number of edge containing y and by the value of stream function ψ on the level component corresponding to y. Denote by $H : R^2 \to \Gamma$ the projection of R^2 on Γ: $H(x_1, x_2)$ is equal to $y \in \Gamma$ corresponding to the level component containing (x_1, x_2); $H(x_1, x_2) = (\psi(x_1, x_2), k(x_1, x_2)) \in \Gamma$, here $k(x_1, x_2)$ is the number of the edge containing y.

Let $\Pi = \Gamma \times R^1$ and define a map $\wedge : R^3 \to \Pi$, $\wedge(x_1, x_2, x_3) = (H(x_1, x_2), x_3) = (\psi(x_1, x_2), k(x_1, x_2), x_3)$ (see [5]). The space Π is called an open book. For example, in the case of the stream function ψ shown in Fig. 2, the space Π is shown in Fig. 3. It has 3 pages and a binding $\{O_2\} \times R^1$.

If the stream function has saddle points, the speed in x_3-direction can be sensitive to small change of ε, and the limit as $\varepsilon \downarrow 0$ may not exist. On the other hand a small noise (small diffusion) is always available in the system, So that we replace the dynamical system (3) governing the motion of the particles by a diffusion process with small diffusion coefficients:

$$\dot{X}_t^{\varepsilon,\kappa}(t) = \frac{1}{\varepsilon}\bar{\nabla}\psi(X_t^{\varepsilon,\kappa}) + b(X_t^{\varepsilon,\kappa}) + \sqrt{\kappa}\sigma\dot{W}_t \tag{7}$$

Here σ is a diagonal matrix with constant non-zero entries, W_t is the three dimensional Wiener process, $0 < \kappa << 1$. The process $X_t^{\varepsilon,\kappa}$ is governed by the differential operator $L^{\varepsilon,\kappa}$, introduced in (2), with $a = \sigma^2$.

We will see that after such a regulazation, there exists a limit of x_3-component of the process $X_t^{\varepsilon,\kappa}$ as $\varepsilon \downarrow 0$. Moreover, if after that we take $\kappa \downarrow 0$, the double limit is independent of the diffusion matrix a. Note that addition of a small diffusion term makes also more regular the domain occupied by the ingredient at time $t > 0$ (under some minor additional assumptions).

The process $X_t^{\varepsilon,\kappa}$ also has a fast and a slow components as $\varepsilon \downarrow 0$. the fast component is again the deterministic motion along the trajectories of the system $\dot{X}_t = \frac{1}{\varepsilon}\bar{\nabla}\psi(X_t)$, and the slow component is the projection $\wedge(X_t^{\varepsilon,\kappa}) = (\psi(X_t^{\varepsilon,\kappa}), k(X_t^{\varepsilon,\kappa}), X_3^{\varepsilon,\kappa}(t))$ of $X_t^{\varepsilon,\kappa}$ on the open book Π.

Let $(z, k) \in \Gamma$, and (z, k) is not an interior vertex. Define

$$A_k(z, y) = \int_{G_k(z)} b_3(x_1, x_2, y)dx_1 dx_2,$$

where $G_k(z) \in R^2$ is the domain bounded by $C_k(z)$. As in the no-saddle-points case, $A_k(z, y) > 0$, $\frac{\partial A_k}{\partial z}(z, y) > 0$, $A_k(z, y) = A_k(z, y + 2\pi)$. If $(H^*, k) = O_2$ is an interior vertex (this means that $H^{-1}(O_2)$ contains a saddle point O_2) (See Fig 2), and $\psi(O_2) = H^*$, the set $G_k(H^*)$ is bounded by an ∞-shaped curve corresponding to O_2. It consists of two parts G_2 and G_3 (see Fig. 3[d]). We put

$$A_2(H^*, y) = \int_{G_2} b_3(x_1, x_2, y)\, dx_1 dx_2, \quad A_3(H^*, y) = \int_{G_3} b_3(x_1, x_2, y)\, dx_1 dx_2,$$
$$A_1(H^*, y) = A_2(H^*, y) + A_3(H^*, y).$$

On each page $I_k \times R^1$, $I_k \subset \Gamma$, define an operator L_k^κ:

$$L_k^\kappa v(z, y) = \frac{1}{T_k(z)}\bar{\nabla} A_k(z, y) \cdot \nabla v + \frac{\kappa}{2T_k(z)}\left[\frac{\partial}{\partial z}\left(D_k^1(z)\frac{\partial}{\partial z}\right) + D_k^2(z)\frac{\partial^2 v}{\partial y^2}\right],$$

The coefficients $A_k(z, y)$ are defined above, and

$$D_k^1(z) = \int_{G_k(z)} \left(a_{11}\frac{\partial^2 \psi(x_1, x_2)}{\partial x_1^2} + a_{22}\frac{\partial^2 \psi(x_1, x_2)}{\partial x_2^2}\right) dx_1 dx_2, \quad D_k^2(z) = a_{33} T_k(z). \tag{8}$$

Let $\wedge_t^\kappa = (Z_t^\kappa, k_t^\kappa, y_t^\kappa)$ be the Markov process on Π governed by the operators L_k^κ inside the pages and satisfying the following gluing conditions on the binding $\{O_2\} \times R^1$ of the open book Π: a continuous on Π and smooth inside the pages up to their boundary function $v_k(z, y)$, $y \in R^1$, $(z, k) \in \Gamma$, belongs to the domain of definition of the generator $\mathbf{A}$ of the process $\wedge_t^\kappa$, if the function $L_k^\kappa v_k(z, y)$ is continuous on Π and satisfies the following condition on the binding $\{O_2\} \times R^1$

$$D_2^1(O_2)\frac{\partial^{(2)} v_2(z, y)}{\partial z} + D_3^1(O_2)\frac{\partial^{(3)} v_3(z, y)}{\partial z} + \left(D_2^1(O_2) + D_3^1(O_2)\right)\frac{\partial^{(1)} v_1(z, y)}{\partial z}\bigg|_{z=\psi(O_2)} = 0,$$

where $\frac{\partial^{(k)} v_k(z,y)}{\partial z}$ is the derivative in z on the page $I_k \times R^1$, $k \in \{1, 2, 3\}$; the coefficients $D_k^1(O_2)$ are defined by formulas (8) with $G_k(z)$ replaced by domains G_k, $k \in \{2, 3\}$, $G_1 = G_2 \cup G_3$, bounded by the ∞-shaped curve corresponding to the vertex O_2 (Fig. 2). If the graph Γ has more than one interior vertex, the gluing conditions on corresponding part of the binding are formulated in a similar way.

One can prove that there exists a unique diffusion process $\wedge_t^\kappa$ on Π governed by the operators L_k^κ and these gluing conditions ([5],[6]). Note that if $b(x) \equiv 0$, such a process has independent components $(z^\kappa(t), k^\kappa(t))$ and $y^\kappa(t)$. The process $(z^\kappa(t), k^\kappa(t))$ is the diffusion process on Γ with corresponding gluing conditions (compare with [6],[4]); $y^\kappa(t)$, in this case, is just an one-dimensional diffusion process. Addition of the term $b(x)$ leads to an absolute continuous change of measures (Girsanov's transformation) in the space of trajectories. This allows to prove existence of the process $\wedge_t^\kappa$ and to calculate the gluing conditions.

It follows from [5], [7] that process $\wedge(X_t^{\varepsilon,\kappa}) = (\psi(X^{\varepsilon,\kappa}(t)), k(X_t^{\varepsilon,\kappa}), X_3^{\varepsilon,\kappa}(t))$ on Π converge weakly in the space of continuous function $[0, T] \to \Pi$ as $\varepsilon \downarrow 0$ to the process $\wedge_t^\kappa$ on Π defined above.

Define now one more stochastic process $\wedge_t = \{z_t, k_t, y_t\}$ on the open book Π: Inside each page $I_k \times R^1 \subset \Pi$, process $\wedge_t$ is deterministic and is defined by equations

$$\dot{z}_t = -\frac{1}{T_k(z_t)}\frac{\partial A_k}{\partial y}(z_t, y_t), \quad \dot{y}_t = \frac{1}{T_k(z_t)}\frac{\partial A_k}{\partial z}(z_t, y_t)$$

where

$$A_k(z, y) = \int_{G_k(z)} b_3(x_1, x_2, y)dx_1dx_2.$$

When the trajectory $\wedge_t$ comes to the binding, say from page 1 on Fig. 3 to a point $(O_2, y) \in \{O_2\} \times R_1$, and $\frac{\partial A_i}{\partial y}(H^*, y) > 0$, $i \in \{2, 3\}$, then $\wedge_t$ goes, without delay and independently of the past, to page 2 or 3 respectively with probabilities

$$P_2(y) = \frac{\frac{\partial A_2}{\partial y}(H^*, y)}{\frac{\partial A_2}{\partial y}(H^*, y) + \frac{\partial A_3}{\partial y}(H^*, y)}, \quad P_3(y) = 1 - P_2(y).$$

It follows form the definition of $A(H^*, y)$ that, if the trajectory $\wedge_t$ comes to a point y of the binding from page 1, then at least one of the derivatives $\frac{\partial A_2}{\partial y}(H^*, y)$ or $\frac{\partial A_3}{\partial y}(H^*, y)$ is positive. If just one of derivatives $\frac{\partial A_i}{\partial y}(H^*, y_t)$, $i \in \{2, 3\}$ is positive $\wedge_t$ goes to the corresponding page with probability 1. These conditions define process $\wedge_t$ in the unique way. The point $(O_2, y) \in \{O_2\} \times R^1$ is uniquely defined by the value of $A_1(H^*, y)$, so that we can consider P_2 and P_3 as functions of A_1: $P_2 = \tilde{P}_2(A_1)$, $P_3 = \tilde{P}(A_1)$.

It follows from [5] that the process $\wedge_t^\kappa$ converge weakly as $\kappa \downarrow 0$ to $\wedge_t$ on any finite time interval $[0, T]$. Note that process $\wedge_t$ independent of the matrix $a = \sigma\sigma^*$. Thus the stochasticity of $\wedge_t$ is an intrinsic property of deterministic flow. The additional diffusion is used just for regularization of the problem.

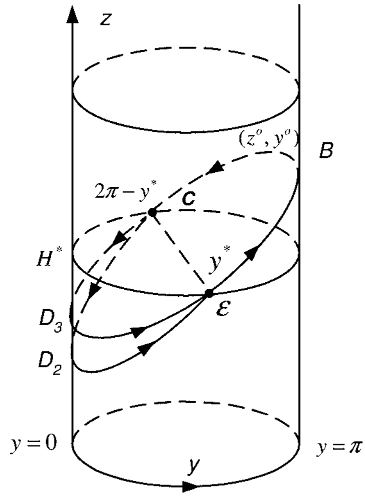

Fig. 4.

Assume that 2π-periodic functions $A_k(H^*, y)$, $k \in \{1, 2, 3\}$ have the following symmetry property: $A_k(H^*, \pi - y) = A_k(H^*, \pi + y)$, where $H^* = \psi(O_2)$ is the value of stream function at the saddle point (Fig 2). Then, of course, functions $A_k(H^*, y)$ have critical points at $y = 0$ and at $y = \pi$. We assume that these functions have no other critical points, and that $y = 0$ is the minimum and $y = \pi$ is the maximum point. We will refer to these properties as Assumption S.

Let the support of the initial function be concentrated near a point $(x_1^0, x_2^0, y^0) \in R^3$, $z^0 = \psi(x_1^0, x_2^0)$, and let, to be specific, $z^0 > H^*$. If $A_1(z^0, y^0) > \max_{0 \le y \le 2\pi} A_1(H^*, y)$, then the particles "do not feel" the saddle point, and the limiting slow motion is as in the no-saddle-point case.

Let now $A_1(z^0, y^0) < \max_{0 \le y \le 2\pi} A_1(H^*, y)$ and the Assumption S be satisfied. Then the limiting slow motion will go along the curve $\gamma_1 = \mathcal{E}BC$ (Fig. 4). The curve γ_1 is defined by equation

$$A_1(z, y) = A_1(z^0, y^0), \quad z \ge H^*.$$

When the slow trajectory comes to the level $z = H^*$ (to the binding of the open book in Fig. 3) it can go to page 2 or 3 with probability $P_2(y^*)$ and $P_3(y^*) = 1 - P_2(y^*)$ respectively. On page $i \in \{2, 3\}$, it moves along the curve γ_i defined by equation

$$A_i(z, y) = A_i(H^*, y^*), \quad z < H^*.$$

The curve γ_2 in Fig. 4 is $CD_2\mathcal{E}$ and γ_3 is $CD_3\mathcal{E}$.

The time to pass the curve γ_i, $i \in \{1, 2, 3\}$, is equal to

$$T_i = \int_{\gamma_i} \frac{T_i(z)dl}{|\nabla A_i(z, y)|}. \tag{9}$$

Because of Assumption S, all three curves $\gamma_1, \gamma_2, \gamma_3$ connect the same points (H^*, y^*) and $(H^*, 2\pi - y^*)$. This means that every interval $[y^0 + 2\pi l, y^0 + 2\pi(l + 1)]$ the trajectory of limiting slow motion goes along $\gamma_1 + \gamma_2$ or along $\gamma_1 + \gamma_3$ with probabilities $P_2(y^*)$ and $P_3(y^*)$ respectively, independently of the motion on other such intervals. Therefore the time to go in x_3-direction from y^0 to $y^0 + 2\pi N$ is equal to the sum of independent identically distributed random variables $\tau_1 + \tau_2 + \ldots + \tau_N$, where

$$\tau_i = \begin{cases} T_2 + T_1, \text{ with probability } P_2(y^*), \\ T_3 + T_1, \text{ with probability } P_3(y^*). \end{cases}$$

The average speed (without reaction term) of the fluid in x_3-direction is, according to the law of large numbers

$$\frac{2\pi N}{\tau_1 + \ldots + \tau_N} = \frac{2\pi}{\frac{1}{N}\sum_1^N \tau_i} \to \frac{2\pi}{E\tau_i}$$
$$= \frac{2\pi}{T_1 + T_2 P_2(y^*) + T_3 P_3(y^*)} = \nu(\psi(x_1^0, x_2^0), y_0^0), \quad as\ N \to \infty.$$

Note that this speed is independent of the diffusion matrix a. If $Suppg = G_0$, the spectrum of asymptotic speeds along x_3-axis is equal to

$$\cup_{(x_1^0, x_2^0, y^0) \in G_0} \nu(\psi(x_1^0, x_2^0), y^0).$$

So that, for large t, the spot is stretching in x_3-direction between $t \cdot \min_{(x_1^0, x_2^0, x_3^0) \in G_0} \nu(\psi(x_1^0, x_2^0), x_3^0)$ and $t \cdot \max_{(x_1^0, x_2^0, x_3^0) \in G_0} \nu(\psi(x_1^0, x_2^0), x_3^0)$. The density of the ingredient will decrease to preserve the total amount of ingredient (in the case $f \equiv 0$).

When a reaction term of KPP-type is included in equation, the spot occupied by ingredient, as a rule, moves faster than without this term, because of multiplication of the particles. This effect is a manifestation of large deviations for sums of independent (if Assumption S is satisfied) random variables. In the case of one saddle point (Fig. 3, 4), the random variables have just two values. In this case the action functional for large deviation asymptotics is equal to the relative entropy (Fig. 5) [2].

$$H(\alpha) = H_{y^*}(\alpha) = \alpha \ln \frac{\alpha}{P_2(y^*)} + (1 - \alpha) \ln \frac{1 - \alpha}{1 - P_2(y^*)}.$$

In this case

$$\lim_{N \to \infty} N^{-1} \ln P\{\text{in N Bernoulli trials with success probability } \mathrm{P}_2(\mathrm{y}^*) \text{ occurs}$$
$$\alpha N\ success\} = -H(\alpha).$$

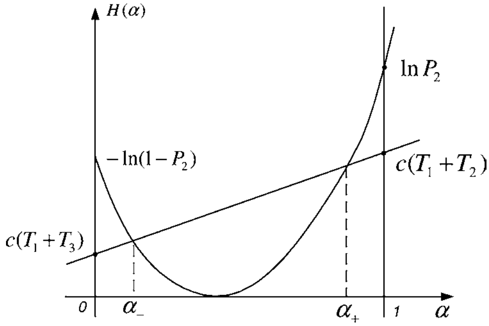

Fig. 5.

It follows from the Feynman-Kac formula that the solution $u^{\varepsilon,\kappa}(t,x)$ of problem (2) satisfies the equation

$$u^{\varepsilon,\kappa}(t,x) = E_x g(X_t^{\varepsilon,\kappa}) \exp\left\{\int_0^t c\big(u^{\varepsilon,\kappa}\big(t-s, X_s^{\varepsilon,\kappa}\big)\big)\, ds\right\}. \tag{10}$$

Consider the equation

$$v(t,z,k,y) = E_{z,k,y}\bar{g}(\wedge_t) \exp\left\{\int_0^t c(v(t-s,\wedge_s))\, ds\right\}, \quad (z,k,y) \in \Pi, \tag{11}$$

$$\bar{g}(z,k,y) = \frac{1}{T_k(z)} \oint_{C_k(z)} \frac{g(x_1,x_2,y)dl}{|\nabla\psi(x_1,x_2)|}.$$

It is easy to check that since $c(v)$ is Lipschitz continuous, equation (11) has a unique solution (compare with [3],Ch. 5). Actually, $v(t,z,k,y)$ is the solution, maybe generalized, of the Cauchy problem

$$\frac{\partial v(t,k,z,y)}{\partial t} = \frac{1}{T_k(z)} \bar{\nabla} A_k(z,y) \cdot \nabla_{z,y} v + f(v).$$

inside the pages of the open book Π, satisfying the gluing conditions on the binding of the book and initial condition $v(0,k,z,y) = \bar{g}(z,k,y)$.

Using convergence of $\wedge(X_t^{\varepsilon,\kappa})$ to $\wedge_t$ as, first, ε and then κ tend to zero, and equations (10), (11), one can prove that $u^{\varepsilon,\kappa}(t,x)$ approaches $v(t,\wedge(x))$ as $\varepsilon.\kappa \downarrow 0$ (compare with [4]). Note that the function $v(t,\wedge(x))$ is independent of the diffusion matrix a.

Since the nonlinear term $f(u) = c(u)u$ is of KPP type, $c(u) \leq c(0) = c$ for $u \geq 0$. Then we derive from (11)that

$$v(t,z,k,y) \leq E_{z,k,y}\bar{g}(\wedge_t)e^{ct}. \tag{12}$$

Let $N = [\frac{y_t - y}{2\pi}]$ be the integer part of $\frac{y_t - y}{2\pi}$. Then the trajectory $\wedge_s = (z_s, k_s, y_s)$, $0 \le s \le t$. visits the binding $\{O_2\} \times R^1$ of the open book Π (Fig. 4) N times. Let $\gamma(N)$ be the number of times among those N, when $\wedge_s$ goes from page 1 to page 2 of Π. Then

$$t = \gamma(N)(T_1 + T_2) + [N - \gamma(N)](T_1 + T_3) + O_N(1), \quad N \to \infty,$$

where T_i, $i \in \{1, 2, 3\}$, are defined by (9).

Let the support G_0 of the initial function $g(x)$, $x \in R^3$ be situated between $x_3 = -m$ and $x_3 = 0$, and let $T_2 > T_3$. Let $\alpha_\pm$, $\alpha_- < \alpha_+$, be the roots of equation (Fig. 5).

$$c\alpha(T_1 + T_2) + c(1 - \alpha)(T_1 + T_3) = H(\alpha). \tag{13}$$

If $T_1 + T_3 > -\ln(1 - P_2)$ put $\alpha_- = 0$, and if $T_1 + T_2 > -\ln P_2$ put $\alpha_+ = 1$. Note that times T_i depend on the initial point $u = (z^o, k^o, y^o)$, therefore $\alpha_\pm = \alpha_\pm(u)$.

Define

$$V_- = \min_{u \in \wedge(G_0)} \frac{2\pi}{\alpha_+(u)(T_1(u) + T_2(u)) + (\alpha_-(u)(T_1(u) + T_3(u))},$$

$$V_+ = \max_{u \in \wedge(G_0)} \frac{2\pi}{\alpha_+(u)(T_1(u) + T_2(u)) + (\alpha_-(u)(T_1(u) + T_3(u))}.$$

Taking into account that the large deviation asymptotics for $\frac{\gamma(N)}{N}$, $N \to \infty$, is described by the action function $NH(\alpha)$, one can conclude that

$$\lim_{t \to \infty} \frac{1}{t} \ln E_{z,k,\beta t}\, \bar{g}(\wedge_t) e^{ct} = -\infty$$

if $\beta \notin [V_-, V_+]$. Then (12) implies

$$\lim_{t \to \infty} \max_{z,k} v(t, z, k, \beta t) = 0, \quad \text{if } \beta \notin [\mathrm{V}_-, \mathrm{V}_+].$$

On the other hand, using arguments similar to arguments used in section 6.2 of [3], we can check that

$$\lim_{t \to \infty} \max_{z,k} v(t, z, k, \beta t) = 1, \quad \text{if } \beta \in (\mathrm{V}_-, \mathrm{V}_+).$$

Since $\lim_{\kappa \downarrow 0} \lim_{\varepsilon \downarrow 0} u^{\varepsilon,\kappa}(t, x) = v(t, \wedge(x))$, we come to the following result:

Theorem 1. *Assume that the stream function $\psi(x)$, $x \in R^2$, has just one saddle point O_2, $\psi(O_2) = H^*$. Let the perturbation $b(x)$, $x \in R^3$, has mentioned above properties, in particular, $div b(x) = 0$, $b(x_1, x_2, x_3) \equiv b(x_1, x_2, x_3 + 2\pi)$, and assumption S is satisfied.*

Let the set $G_0 = Supp.g$ be connected , and the closure $[G_0]$ of G_0 coincides with the closure of its interior (G_0): $[G_0] = [(G_0)]$. Assume *that* $G_0 \subset \{(x_1, x_2, x_3) : x_3 \in [-m, 0]\}$*;* $z^0 = \psi(x_1^0, x_2^0) > H^*$*,* $A_1(\wedge(x^0)) < \max_{0 \le y \le 2\pi} A_1(H^*, y)$*,* $T_2(A_1(\wedge(x^0))) > T_3(A_1(\wedge(x^0)))$ *for* $x^0 \in G_0$.

Let $f(u) = c(u)u$ *be of KPP-type, and* $c = c(0)$*, then*

$$\lim_{t\to\infty} \lim_{\kappa\downarrow 0} \lim_{\varepsilon\downarrow 0} \max_{x_1,x_2} u^{\varepsilon,\kappa}(t, x_1, x_2, \nu t) = \begin{cases} 1, \text{ if } \nu \in (V_-, V_+), \\ 0, \text{ if } \nu \notin [V_-, V_+]. \end{cases}$$

In particular, if the ingredient is concentrated in a small neighborhood of a point $x^0 \in R^3$, the asymptotic speed of the ingredient particles in x_3-direction without reaction is close to

$$\bar{\nu} = \frac{2\pi}{P_2(T_1 + T_2) + (1 - P_2)(T_1 + T_3)},$$

where P_2 and T_i are calculated for $A_1 = A_1(\wedge(x^0))$. The reaction term accelerates the motion of the area occupied by particles and lead to stretching of this area in x_3-direction: the spectrum of speeds consists now of a small neighborhood of the interval

$$\left[\frac{2\pi}{\alpha_+(T_1 + T_2) + (1 - \alpha_+)(T_1 + T_3)}, \frac{2\pi}{\alpha_-(T_1 + T_2) + (1 - \alpha_-)(T_1 + T_3)}\right],$$

where $\alpha_\pm$ and T_i are calculated for $A_1 = A_1(\wedge(x^0))$. Of course, $\bar{\nu}$ belongs to this interval. The spectrum of speeds becomes wider and wider as $c = c(0)$ increases up to $\max(\ln P_2^{-1}, \ln(1 - P_2)^{-1})$. For c greater than this maximum, the interval is equal to $[\frac{2\pi}{T_1+T_2}, \frac{2\pi}{T_1+T_3}]$.

We will finish this section with short remarks on general case, when Assumption S is not satisfied. As before, we assume that the stream function has one saddle point O_2, $z^0 = \psi(x_1^0, x_2^0) > H^* = \psi(O_2)$, and $A_1(\wedge(x_0)) < \max_{0<y\le 2\pi} A_1(H^*, y)$.

Consider the circle $\{z = H^*, 0 \le y < 2\pi\} = \mathcal{E}$ (Fig. 6). Define continuous mappings of $\mathcal{E}$ in itself. The mapping $F_i : \mathcal{E} \to \mathcal{E}$, $i \in \{2, 3\}$ is defined as follows: Consider the trajectory (z_t, y_t) on the cylinder $\mathcal{E} \times R^1$. First, the trajectory starting at (z_0, y_0) comes to a point $M_0 \in \mathcal{E}$ (see Fig. 6) along the curve $\{A_1(z, y) = A_1(z_0, y_0)\}$. Then it goes along the curve γ_i defined by equation $A_i(z, y) = A_1(z_0, y_0)$, $i \in \{2, 3\}$, until it comes back to $\mathcal{E}$ at a point $B_i(M_0)$. Then it goes from $B_i(M_0)$ along the curve $A_1(z, y) = A_1(H^*, B_i(M_0))$ to a point M_1^i of $\mathcal{E}$, where the trajectory again cross the circle $\mathcal{E}$. The map $M_0 \to M_1^i$, $i \in \{2, 3\}$, is called $F_i(M_0)$. Without Assumption S, $F_i(M_0)$, in general, differs from M_0. Each time when the trajectory comes from above to $\mathcal{E}$ at a point M, it goes along γ_2 or γ_3 with probabilities $P_2(M)$ and $1 - P_2(M)$ respectively, independently of the past behavior. So that from M trajectory goes to $F_2(M)$ or $F_3(M)$ with probabilities $P_2(M)$ and $1 - P_2(M)$. So that we have a Markov chain on $\mathcal{E}$: in one step the

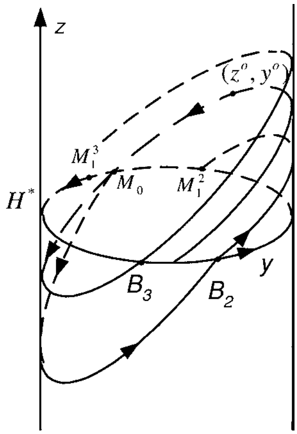

Fig. 6.

chain jumps from $M \in \mathcal{E}$ to $F_2(M)$ or to $F_3(M)$ with probabilities $P_2(M)$ and $1 - P_2(M)$ respectively. Let $T_i(M)$, $i \in \{2, 3\}$ the time which trajectory spends for transition from M to $F_i(M)$ if it uses curve γ_i. Under some natural additional assumptions, the chain is ergodic and has a unique invariant measure $\lambda(dy)$ on $\mathcal{E}$. Then the asymptotic speed of the spot occupies by the ingredient in x_3-direction is independent of the initial position of the spot and is equal to

$$\bar{\nu} = \frac{2\pi}{\int_0^{2\pi} [P_2(y)T_2(y) + (1 - P_2(y))T_3(y)]\lambda(dy)}.$$

Some sufficient conditions for ergodicity of the chain, one can derive from [8]. The asymptotic speed of the ingredient when a KPP-type reaction term is included in the equation can be described by the relative entropy for this chain, We are going to consider these questions elsewhere.

4. PROPAGATION OF THE INGREDIENT IN A LARGER TIME SCALE

Until now we considered evolution of the domain occupied by the ingredient as $0 < \varepsilon, \kappa << 1$ in a large but finite time interval. Then the spot occupied by the ingredient grows and moves in x_3-direction and stays bounded in (x_1, x_2)-plane. The evolution of the spot is independent of the diffusion matrix (a_{ij}).

Now, we consider the question of growth in (x_1, x_2)-plane. This growth occurs when t tends to infinity as $\kappa^{-1/2}$. For brevity, we restrict ourselves to the case of stream function $\psi(x)$ with just one critical point-minimum at the origin O, $\psi(O) = 0$. The nonlinear term, as before, is of KPP-type. We assume that the support G_0 of the initial density $g(x)$, $x \in R^3$, is invariant with respect to shifts along x_3-axis: $G_0 = \tilde{G} \times R^1$, where $\tilde{G}_0$ is a bounded connected domain in (x_1, x_2)-plane. Let $\bar{G}_0 = \{(z, y), y \in R^1, z = \psi(x_1, x_2)$ for some $(x_1, x_2) \in \tilde{G}\}$.

One can prove (compare with [4])that under our assumptions $\lim_{\varepsilon\downarrow 0} u^{\varepsilon,\kappa}(t, x) = \nu^\kappa(t, \wedge(x))$ exists for $t > 0$, and $\nu^\kappa(t, z, y)$, $z > 0$, $y \in R^1$, is the solution of the problem

$$\frac{\partial \nu^\kappa(t,z,y)}{\partial t} = \frac{\kappa}{2T(z)}\left(D^1(z)\frac{\partial^2 \nu^\kappa}{\partial z^2} + D^2(z)\frac{\partial^2\nu^\kappa}{\partial y^2}\right) + \frac{1}{T(z)}\bar{\nabla} A(z, y)\cdot\nabla\nu^\kappa$$
$$+ f(\nu^\kappa) = L^\kappa \nu^\kappa + f(\nu^\kappa), \quad \nu^\kappa(0, z, y) = \bar{g}(z, y) \tag{14}$$

The coefficients $D^1(z)$, $D^2(z)$ in (14) and $A(z, y)$ were defined in the previous section. Note that since we assume that $\psi(x)$ has no saddle points, the open book Π consists of one page in this case.

Let $\tilde{\nu}^\kappa(t, z, y) = \nu(\frac{t}{\sqrt{\kappa}}, z, y)$. Then $\tilde{\nu}^\kappa(t, z, y)$ satisfies to equation

$$\frac{\partial\tilde{\nu}^\kappa}{\partial t} = \frac{1}{\sqrt{\kappa}}L^\kappa\tilde{\nu}^\kappa + \frac{1}{\sqrt{\kappa}}f(\tilde{\nu}^\kappa), \quad \tilde{\nu}^\kappa(0, z, y) = \bar{g}(z, y)$$

The diffusion process $\tilde{\wedge}_t^\kappa$, corresponding to the operator $\frac{1}{\sqrt{\kappa}}L^\kappa$, has a fast component, which is the motion along the levels curve of $A(z, y)$, and a slow component, which is transversal to the level curves.

As it follows from the Feynman-Kac formula, the function $\tilde{\nu}^\kappa(t, z, y)$ is the solution of the equation:

$$\tilde{\nu}^\kappa(t, z, y) = E_{z,y}\bar{g}(\tilde{\wedge}_t^\kappa)\exp\left\{\frac{1}{\sqrt{\kappa}}\int_0^t c(\tilde{\nu}^\kappa(t-s, \tilde{\wedge}_s^\kappa))ds\right\}. \tag{15}$$

Since $c(\nu) \le c(0)$, $\tilde{\wedge}^\kappa(t, z, y) \le E_{z,y}\bar{g}(\tilde{\wedge}_t^\kappa)\exp\{\frac{ct}{\sqrt{\kappa}}\}$.

Consider the closed curve $\sum_m = \{(z, y) : A(z, y) = m,\ 0 \le y < 2\pi\}$ (Fig. 1); One rotation along $\sum_m$ takes time

$$\tau_m = \oint_{\sum_m}\frac{T(z)dl}{|\nabla A(z, y)|}.$$

Put $\bar{\bar{G}}_0 = \{a \in R^1 : A(z, y) = a$ for some $(z, y) \in \bar{G}_0\}$. If $\frac{d\tau_m}{dm} \ne 0$ for $m \in \bar{\bar{G}}_0$, then the area where $\tilde{\nu}^\kappa(t, z, y)$ is close to 1, for any $t > 0$ and $\kappa \downarrow 0$, is invariant with respect to shifts along the vector field $\bar{\nabla} A(z, y)$. This can be checked using same arguments as in Theorem 3.1 from [3]. Similarly to [4], one can also show that the probabilities of large deviations for the process $A(\wedge_t^\kappa)$ as $\kappa \downarrow 0$ are the same as for the diffusion process $\bar{\bar{X}}_t^\kappa$ governed by the operator $\bar{\bar{L}}^\kappa = \frac{\sqrt{\kappa}}{2}\bar{\bar{D}}(m)\frac{d^2}{dm^2}$, where

$$\bar{\bar{D}}(m) = \frac{1}{\tau_m}\oint_{\sum_m}\frac{(D(z)\nabla A(z, y)\cdot\nabla A(z, y))dl}{T(z)|\nabla A(z, y)|},$$

$D(z)$ is the diagonal 2×2 matrix with the entries $D^1(z)$ and $D^2(z)$. The action functional for the process $\bar{\bar{X}}_t^\kappa$, $0 \le t \le T$, as $\kappa \downarrow 0$ in the space C_{0T} of continuous functions on $[0, T]$ has the form

$$\frac{1}{\kappa} S_{0T}(\varphi) = \begin{cases} \frac{1}{2\kappa} \int_0^T \frac{\dot{\varphi}_s^2 ds}{\bar{\bar{D}}(\varphi_s)}, & \text{if } \varphi \text{ is absolutely continuous and } \varphi_0 = \bar{\bar{X}}_0; \\ +\infty, & \text{for the rest of } C_{0T}. \end{cases}$$

Then the standard arguments (see Ch.6 in [3]) lead to the following result.

Theorem 2. *Let the stream function $\psi(x)$, $x \in R^2$, have just one critical point-minimum at the origin, $\lim_{|x| \to \infty} \psi(x) = \infty$. Let $Supp. g = \tilde{G}_0 \times \{-\infty, \infty\}$, where $\tilde{G}_0$ is a connected bounded domain in the (x_1, x_2)-plane, $\bar{\bar{G}}_0 = \{a \in R^1 : A(\psi(x_1, x_2), x_3) = a$ for some $(x_1, x_2) \in \tilde{G}, x_3 \in R^1\}$. Let $b(x)$, $x \in R^3$, be 2π-periodic in x_3, $div\, b(x) = 0$, and $b_3(x) \ge \bar{b} > 0$. Assume that $\frac{dT(z)}{dz} \neq 0$ for $z = \psi(x_1, x_2)$, $(x_1, x_2) \in G_0$, and $\frac{d\tau_m}{dm} \neq 0$ for $m \in \bar{\bar{G}}_0$. Introduce a distance $\rho(\cdot, \cdot)$ in R^1:*

$$\rho(h_1, h_2) = \left| \int_{h_1}^{h_2} \left[\bar{\bar{D}}(m) \right]^{-\frac{1}{2}} dm \right|.$$

Then

$$\lim_{\kappa \downarrow 0} \lim_{\varepsilon \downarrow 0} u^{\varepsilon,\kappa}(\frac{t}{\sqrt{\kappa}}, x) = \begin{cases} 1, & \text{if } \rho(A(\wedge(x)), \bar{\bar{G}}_0) < t\sqrt{2c}; \\ 0, & \text{if } \rho(A(\wedge(x)), \bar{\bar{G}}_0) > t\sqrt{2c}. \end{cases}$$

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